§7.4 Confidence Intervals for Variance and Standard Deviation of Normal Population.

Suppose X is Normally distributed and $X_1, X_2, \dots, X_n$ are IID samples of X.

$$S^2 = \frac{(X_1-\bar{X})^2 + \cdots + (X_n-\bar{X})^2}{n-1} = \frac{1}{n-1}\left(\sum_1^n X_k^2 - \frac{1}{n}\left(\sum_1^n X_k\right)^2\right)$$

is an unbiased estimator for σ^2.

Confidence interval ?

Thm: If X is Normal & $X_1, X_2, \dots X_n$ are IID samples then

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_1^n (X_k-\bar{X})^2}{\sigma^2}$$

is distributed as $\chi^2(n-1)$

"Chi-Squared with $n-1$ degrees of freedom"

Recall: $\chi^2(n) = \text{Gamma}\left(\frac{n}{2}, 2\right)$

Sum of squares of IID Normal(0,1) r.v. is $\chi^2(n)$

[Write $\chi^2_{\alpha,n}$ for critical value with

$$P(X^2 > \chi^2_{\alpha,n}) = \alpha$$

where $X^2 \sim \chi^2(n)$.]

(1-α) Confidence Interval

$$P\left(\chi^2_{1-\alpha/2,\,n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2,\,n-1}\right) = 1-\alpha$$

solve for σ^2

$$\boxed{\frac{(n-1)S^2}{\chi^2_{\alpha/2,\,n-1}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,\,n-1}}}$$

take $\sqrt{\ }$

$$\boxed{S\sqrt{\frac{n-1}{\chi^2_{\alpha/2,\,n-1}}} < \sigma < S\sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\,n-1}}}}$$

In R this is

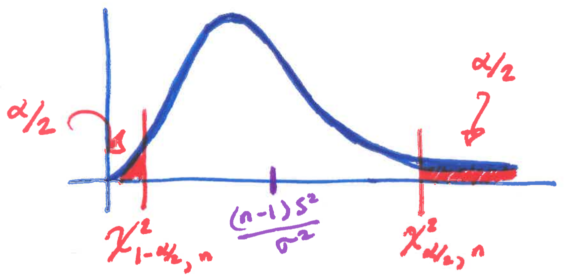
"$S\sqrt{(n-1)/\text{qchisq}(1-\alpha/2,\, n-1)}$"

"$S\sqrt{(n-1)/\text{qchisq}(\alpha/2,\, n-1)}$"

Note: We cannot use the nice

$$\theta \approx \hat{\theta} \pm z_{\alpha/2}\, \sigma_{\hat{\theta}}$$

"$\pm$ style" of notation for confidence interval because χ^2 is not symmetric



Note:

- Mean of $\chi^2(n)$ distribution is n
- Variance is $2n$